X Test-4

Time : 45 mins

1) obtain all zeroes of $p(x) = 8x^4 + 8x^3 - 18x^2 - 20x - 5$
(4) having two of its zeroes are $\sqrt{\frac{5}{2}}$ and $-\sqrt{\frac{5}{2}}$

2) If the polynomial $x^4 - 6x^3 + 16x^2 - 25x + 10$ is divided by
(4) another polynomial $x^2 - 2x + k$, the remainder comes out to be $x + a$, then find k and a.

3) If two zeroes of the polynomial $x^4 - 6x^3 - 26x^2 + 138 - 35$
(4) are $2 \pm \sqrt{3}$, then find other zeroes.

4) If α and β are the zeroes of $f(x) = x^2 - 3x - 2$, find a
(4) polynomial whose zeroes are $\frac{1}{2\alpha+\beta}$ and $\frac{1}{2\beta+\alpha}$.

5) Prove that $2+\sqrt{5}$ is an irrational number

X Test - 4 (Solutions)

1) Since $\sqrt{\frac{5}{2}}$ and $-\sqrt{\frac{5}{2}}$ are zeroes of $p(x)$, then

$\left(x-\sqrt{\frac{5}{2}}\right)$ and $\left(x+\sqrt{\frac{5}{2}}\right)$ are factors of $p(x)$.

Also, $x^2-\frac{5}{2} = \frac{2x^2-5}{2}$ is a factor of $p(x)$

On dividing $p(x)$ by $2x^2-5$,

$$
\begin{array}{r|l}
 & 4x^2+4x+1 \\
\hline
2x^2-5 & 8x^4+8x^3-18x^2-20x-5 \\
 & \underline{\overset{(-)}{}8x^4+0x^3\overset{(+)}{-}20x^2} \\
 & \quad 8x^3+2x^2-20x-5 \\
 & \quad \underline{\overset{(-)}{}8x^3+0x^2\overset{(+)}{-}20x} \\
 & \qquad 2x^2-5 \\
 & \qquad \underline{\overset{(-)}{}2x^2\overset{(+)}{-}5} \\
 & \qquad\quad \underline{\underline{0}}
\end{array}
$$

Using division algorithm,

$$p(x) = (2x^2-5)(4x^2+4x+1)+0$$
$$= (2x^2-5)(2x+1)^2$$
$$= (2x^2-5)(2x+1)(2x+1)$$

$\therefore$ All zeroes of $p(x)$ are $\sqrt{\frac{5}{2}}, -\sqrt{\frac{5}{2}}, \frac{-1}{2}$ and $-\frac{1}{2}$

2) Let $p(x) = x^4-6x^3+16x^2-25x+10$

On dividing $p(x)$ by x^2-2x+k,

$$
\begin{array}{r|l}
 & x^2-4x+(8-k) \\
\hline
x^2-2x+k & x^4-6x^3+16x^2-25x+10 \\
 & \underline{\overset{(-)}{}x^4\overset{(+)}{-}2x^3\overset{(-)}{+}kx^2} \\
 & \quad -4x^3+x^2(16-k)-25x+10 \\
 & \quad \underline{\overset{(+)}{-}4x^3\overset{(-)}{+}8x^2\quad\overset{(+)}{-}4kx} \\
 & \qquad x^2(8-k)+x(4k-25)+10 \\
 & \qquad \underline{\overset{(-)}{}x^2(8-k)\overset{(+)}{-}x(16-2k)\overset{(-)}{+}k(8-k)} \\
 & \qquad x(2k-9)+\underline{\underline{(k^2-8k+10)}}
\end{array}
$$

On comparing the remainder with $x+a$,

$2k-9=1$

$2k=10 \Rightarrow \boxed{k=5}$

$k^2-8k+10=a$

$a = 25-40+10 = -15+10$

$\boxed{a=-5}$

3) Let $p(x) = x^4 - 6x^3 - 26x^2 + 138x - 35$

Since $2+\sqrt{3}$, $2-\sqrt{3}$ are zeroes of $p(x)$, then $(x-(2+\sqrt{3}))$ and $(x-(2-\sqrt{3}))$ are factors of $p(x)$.

Also $[(x-2)-\sqrt{3}]$ and $[(x-2)+\sqrt{3}]$ are also factors of $p(x)$.

Thus $(x-2)^2 - (\sqrt{3})^2 = x^2+4-4x-3 = x^2-4x+1$ is also a factor of $p(x)$.

On dividing $p(x)$ by x^2-4x+1,

$$
\begin{array}{r|l}
 & x^2 - 2x - 35 \\
\hline
x^2-4x+1 & x^4 - 6x^3 - 26x^2 + 138x - 35 \\
 & \underset{(-)}{\cancel{x^4}} \underset{(+)}{-} 4x^3 \underset{(-)}{+} x^2 \\
\hline
 & -2x^3 - 27x^2 + 138x - 35 \\
 & \underset{(+)}{-} \cancel{2x^3} \underset{(-)}{+} 8x^2 \underset{(+)}{-} 2x \\
\hline
 & -35x^2 + 140x - 35 \\
 & \underset{(+)}{-} \cancel{35x^2} \underset{(-)}{+} \cancel{140x} \underset{(+)}{-} \cancel{35} \\
\hline
 & 0
\end{array}
$$

Using division algorithm,

$p(x) = (x^2-4x+1)(x^2-2x-35) + 0$

$= (x^2-4x+1)(x-7)(x+5)$

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$-2 \quad -35 < \begin{matrix} -7 \\ 5 \end{matrix}$

$\therefore$ The other zeroes of $p(x)$ are 7 and -5.

4) Let $f(x) = x^2 - 3x - 2$ be of the form ax^2+bx+c; where $a=1$, $b=-3$, $c=-2$.

$\alpha + \beta = \frac{-b}{a} = \underline{\underline{3}}$

$\alpha\beta = \frac{c}{a} = \underline{\underline{-2}}$

For new polynomial:

Sum of zeroes $= \frac{1}{2\alpha+\beta} + \frac{1}{2\beta+\alpha} = \frac{2\beta+\alpha+2\alpha+\beta}{(2\alpha+\beta)(2\beta+\alpha)} = \frac{3\beta+3\alpha}{4\alpha\beta+2\alpha^2+2\beta^2+\alpha\beta}$

$= \frac{3(\alpha+\beta)}{5\alpha\beta+2[(\alpha+\beta)^2-2\alpha\beta]} = \frac{3\times 3}{5\times -2+2[3^2+2\times 2]}$

$= \frac{9}{-10+2\times 13} = \frac{9}{-10+26} = \underline{\underline{\frac{9}{16}}}$

Product of zeroes, $\dfrac{1}{2\alpha+\beta} \times \dfrac{1}{2\beta+\alpha} = \dfrac{1}{4\alpha\beta+2\alpha^2+2\beta^2+\alpha\beta} = \underline{\underline{\dfrac{1}{16}}}$

$\therefore$ The required polynomial is $x^2 - (\text{sum of zeroes})x + \text{product of zeroes}$

$$= x^2 - \frac{9}{16}x + \frac{1}{16}$$

$$= \frac{1}{16}(16x^2 - 9x + 1) \text{ or } 16x^2 - 9x + 1 \;//$$

5) Let us assume $\sqrt{5}$ is a rational number.
Then $\sqrt{5} = \frac{p}{q}$; where p and q are co-prime numbers and $q \neq 0$.

$$\Rightarrow (\sqrt{5})^2 = \frac{p^2}{q^2}$$

$$\Rightarrow 5q^2 = p^2 \Rightarrow 5 \text{ divides } p^2$$
$$\Rightarrow 5 \text{ divides } p$$

Let $p = 5c$, where c is some integer.
Then $5q^2 = (5c)^2$
$$\Rightarrow 5q^2 = 25c^2$$
$$\Rightarrow q^2 = 5c^2$$
$$\Rightarrow 5 \text{ divides } q^2$$
$\Rightarrow$ 5 divides q. Thus 5 is a common factor for p and q. But this contradicts the fact that p and q are coprimes.
Thus our assumption is wrong and $\sqrt{5}$ is an irrational number.

Let us assume $2+\sqrt{5}$ is a rational number.

Then $2+\sqrt{5} = \frac{a}{b}$; where a and b are integers and $b \neq 0$

$\Rightarrow \sqrt{5} = \frac{a}{b} - 2 = \frac{a-2b}{b}$, which is a rational number since a and b are integers.

Thus $\sqrt{5}$ is also a rational number. But this contradicts the fact that $\sqrt{5}$ is irrational (proved above). This contradiction arises due to our wrong assumption that $2+\sqrt{5}$ is rational. Hence $2+\sqrt{5}$ is irrational.